Equilibrium of a confined, randomly accelerated, inelastic particle: Is there inelastic collapse?

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We consider the one-dimensional motion of a particle randomly accelerated by Gaussian white noise on the line segment $0<x<1$. The reflections of the particle from the boundaries at $x=0$ and 1 are inelastic, with velocities just after and before reflection related by $v_f=-rv_i$. Cornell *et al.* have predicted that the particle undergoes inelastic collapse for $r<r_c=e^{-\pi/\sqrt{3}}=0.163$, coming to rest at the boundary after an infinite number of collisions in a finite time and remaining there. This has been questioned by Florencio *et al.* and Anton on the basis of simulations. We have solved the Fokker-Planck equation satisfied by the equilibrium distribution function $P(x,v)$ with a combination of exact analytical and numerical methods. Throughout the interval $0<r<1$, $P(x,v)$ remains extended, as opposed to collapsed. There is no transition in which $P(x,v)$ collapses onto the boundaries. However, for $r<r_c$ the equilibrium boundary collision rate is infinite, as predicted by Cornell *et al.*, and all moments $|v|^q$, $q>0$ of the velocity just after reflection from the boundary vanish.

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I. INTRODUCTION

Consider a particle randomly accelerated on the line segment $0<x<1$ according to

$$\frac{d^2x}{dt^2}=\eta(t), \quad \langle \eta(t)\eta(t')\rangle = 2\delta(t-t'), \tag{1}$$

where $\eta(t)$ is uncorrelated white noise with zero mean. If the collisions of the particle with the boundaries at $x=0$ and 1 are elastic, the mean-square velocity increases according to

$$\langle v(t)^2\rangle = \langle v(0)^2\rangle + 2t, \tag{2}$$

just as in the absence of boundaries.

In this paper, we assume that the boundary collisions of the randomly accelerated particle are *inelastic*. The velocities just after and before reflection satisfy

$$v_f=-rv_i, \tag{3}$$

where r is the coefficient of restitution. This simple model is of interest in connection with the statistics of driven granular media, where particles tend to cluster, due to inelastic collisions, even though no attractive forces are present. The model was studied by Cornell, Swift, and Bray (CSB) [1], who argued that the particle undergoes "inelastic collapse," i.e., makes an infinite number of collisions in a finite time, comes to rest at the boundary, and remains there, if the coefficient of restitution r is less than the critical value

$$r_c=e^{-\pi/\sqrt{3}}=0.163\cdots. \tag{4}$$

The prediction of inelastic collapse was questioned by Florencio *et al.* [2], who carried out simulations and found that the particle did not adhere to the boundary for any r. Anton [3] reported that his simulations are consistent with an infinite collision rate for $r<r_c$ but also incompatible with localization of the particle at the boundary.

According to Eqs. (2) and (3), the kinetic energy of the randomly accelerated particle increases in between boundary collisions but decreases, for $r<1$, in the collisions. Eventually an equilibrium is reached. Burkhardt, Franklin, and Gawronski (BFG) [4] analyzed the equilibrium distribution $P(x,v)$ for the position and velocity of the particle for $r_c<r<1$. This function satisfies the steady-state Fokker-Planck equation

$$\left(v\frac{\partial}{\partial x}-\frac{\partial^2}{\partial v^2}\right)P(x,v)=0, \tag{5}$$

with the boundary conditions

$$P(x,v)=P(1-x,-v), \tag{6}$$

$$P(0,-v)=r^2P(0,rv), \quad v>0, \tag{7}$$

corresponding to reflection symmetry and conservation of probability, respectively. In particular, the second boundary condition ensures that the incident and reflected probability currents at the boundary have equal magnitude

$$I=\int_0^\infty dv\; vP(0,-v)=\int_0^\infty dv\; vP(0,v). \tag{8}$$

Making use of an exact Green's-function solution of Eqs. (5)–(7), BFG found that the boundary collision rate I, defined by Eq. (8), diverges as r approaches r_c from above and that $P(x,v)$ is extended, as opposed to collapsed, at $r=r_c$. In this approach, $P(x,v)$ is obtained as the difference of two integrals, both of which diverge for $r\leqslant r_c$. This was noted by BFG, who, however, incorrectly concluded that the solution to the Fokker-Planck equation breaks down for $r<r_c$.

In this paper, the calculation of $P(x,v)$ is extended to $r<r_c$. In Sec. II, the approach of BFG is reviewed. The divergences, for $r\leqslant r_c$, of the two integrals which determine $P(x,v)$ are shown to cancel, leaving a finite result. Throughout the entire interval $0<r<1$, $P(x,v)$ varies smoothly and analytically with r. There is no transition in which $P(x,v)$ collapses onto the boundaries.

In Sec. III, the equilibrium boundary collision rate is calculated from the results of Sec. II. The collision rate is finite for $r > r_c$ and infinite for $r \leqslant r_c$, as predicted by CSB. All the equilibrium moments $\overline{|v|^q}$, $q > 0$ of the velocity just after striking the boundary [5] vanish for $r \leqslant r_c$.

Our conclusions are summarized in Sec. IV, and some earlier results on inelastic collapse are reexamined.

II. SOLUTION OF THE FOKKER-PLANCK EQUATION

We begin with a brief review of the approach [4] of BFG. Generalizing earlier work of Masoliver and Porrà [6], they showed that the Fokker-Planck equation (5), with reflection symmetry (6), has the exact solution

$$P(x,v) = \int_0^\infty du\ u\ G(x,v,u)P(0,u) \tag{9}$$

for $v > 0$ in terms of the Green's function

$$G(x,v,u) = \frac{v^{1/2}u^{1/2}}{3x}e^{-(v^3+u^3)/9x}I_{-1/3}\left(\frac{2v^{3/2}u^{3/2}}{9x}\right) - \frac{1}{3^{1/3}\Gamma\left(\frac{2}{3}\right)}\int_0^x dy \frac{e^{-v^3/9(x-y)}}{(x-y)^{2/3}} \times [R(y,u) - R(1-y,u)], \tag{10}$$

where

$$R(y,u) = \frac{1}{3^{5/6}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}\frac{u^{1/2}e^{-u^3/9y}}{y^{7/6}(1-y)^{1/6}} \times {}_1F_1\left(-\frac{1}{6},\frac{5}{6},\frac{u^3(1-y)}{9y}\right), \tag{11}$$

and ${}_1F_1(a,b,z)$ is a standard confluent hypergeometric function [7].

To calculate $P(x,v)$ from Eq. (9), one must first determine the unknown function $P(0,u)$ on the right-hand side. Setting $x=1$ in Eqs. (9)–(11) and using $r^2P(0,rv) = P(1,v)$, as follows from Eqs. (6) and (7), leads to the integral equation

$$r^2P(0,rv) = \int_0^\infty du\ u\ G(1,v,u)P(0,u) \tag{12}$$

for $P(0,v)$, where

$$G(1,v,u) = \frac{1}{6\pi}v^{1/2}u^{1/2}e^{-(v^3+u^3)/9} \times \left[\frac{9}{v^3+u^3} + 6\,{}_1F_2\left(1;\frac{5}{6},\frac{7}{6};\frac{v^3u^3}{81}\right)\right], \tag{13}$$

and ${}_1F_2(a;b,c;z)$ is a generalized hypergeometric function [7]. The quantity $vG(1,v,u)$ is of interest in its own right. As discussed in the Appendix, it generalizes McKean's result [8] for the velocity distribution at first return to the boundary from the half–line $x > 0$ to the line segment $0 < x < 1$.

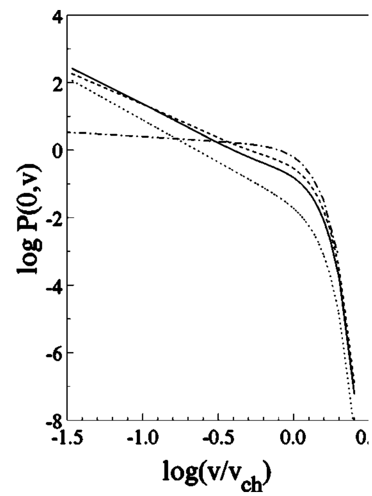

FIG. 1. Double-logarithmic plot (base 10) of $P(0,v)$ for $r=0.01$ (dotted curve), $r=0.1$ (solid curve), $r=0.2$ (dashed curve), and $r=0.5$ (dot-dashed curve). The curves are normalized according to Eq. (20).

BFG showed [4] that the asymptotic form of $P(0,v)$ for small and large v is determined by the first and second terms, respectively, of the kernel $G(1,v,u)$ in Eqs. (12) and (13) and is given by

$$P(0,v) \sim \begin{cases} v^{-\beta(r)}, & v \to 0, \\ e^{-v^3/v_{\mathrm{ch}}(r)^3}, & v \to \infty, \end{cases} \tag{14}$$

where

$$r = \left[2\,\sin\left(\frac{2\beta+1}{6}\pi\right)\right]^{1/(\beta-2)}, \tag{15}$$

$$v_{\mathrm{ch}}(r)^3 = \frac{9r^3}{1-r^3}. \tag{16}$$

Note the non-Maxwellian velocity distribution. As r decreases, the boundary collisions become more inelastic, and the probability of finding the particle near the boundary with a small velocity increases. This is seen in the monotonic increase of the exponent $\beta(r)$ from 0 to $\frac{5}{2}$, as r decreases from 1 to 0. The characteristic velocity $v_{\mathrm{ch}}(r)$ also decreases with decreasing r.

The asymptotic forms (14)–(16) are smooth analytic functions of r throughout the interval $0 < r < 1$. There is no singular behavior at r_c. In particular, on expanding the right side of Eq. (15) about $\beta = 2$, one sees that $\beta(r)$ is a nonsingular function of r, with $\beta(r_c) = 2$. To connect the asymptotic forms (14)–(16) of $P(0,v)$ for small and large v, we have solved the integral equation (12) by numerical iteration, as in [4]. As noted by BFG [4], the integral equation appears to have a well-defined solution for $0 < r < 1$, i.e., $0 < \beta < 5/2$, with no special behavior at r_c. Numerical results for several values of r above and below $r_c = 0.163$ are shown in Fig. 1. The slopes of the curves, for small v, depend on r in accordance with the asymptotic form (14), (15). There is no qualitative difference above and below r_c. Presumably, $P(0,v)$, like its exact asymptotic forms (14)–(16) for small and large v, is an analytic function of r throughout the interval $0<r<1$.

Once $P(0,v)$ has been determined, $P(x,v)$ may be obtained by integration. According to Eqs. (9)–(11), $P(x,v)$ is the sum of two integrals over $P(0,u)$, corresponding to the two terms in the Green's function $G(x,v,u)$ in Eq. (10). Both integrals diverge at the lower limit for $r \leqslant r_c$, as follows from the asymptotic form (14), (15) of $P(0,u)$ for small u, with $\beta(r)>2$ for $r<r_c$. This was noted by BFG [4], who, however, incorrectly concluded that the solution to the Fokker–Planck equation breaks down for $r<r_c$. The divergences cancel, leaving a finite result, as may be seen by integrating with a low-u cutoff and sending the cutoff to zero after adding the two integrals. No cutoff is needed if the two integrands are added before integrating over u. From Eq. (10) it is straightforward to show that $G(x,v,u) \sim u^{1/2}$ in the small-u limit [9]. Thus the integral in Eq. (9) behaves as $\int_0 du\ u^{3/2-\beta}$ for small u. Since $0<\beta(r)<\frac{5}{2}$ for $0<r<1$, there are no convergence problems at the lower limit of the integral. Throughout the interval $0<r<1$, $P(x,v)$ is a smooth well-defined function of r, presumably analytic in r, and does not collapse onto the boundaries at $x=0$ and $x=1$.

We have also considered the probability density

$$P(x) = \int_{-\infty}^{\infty} dv\ P(x,v) = \int_0^{\infty} dv [P(x,v) + P(1-x,v)] \tag{17}$$

for the position of the particle. From Eqs. (9)–(11) and (17), one finds that the leading singular contribution to $P(x)$ for $x \to 0$ is determined by the asymptotic form $P(0,v) \approx A v^{-\beta}$ for $v \to 0$ in Eq. (14) and given by

$$P_{\rm sing}(x) \approx B x^{(1-\beta)/3}, \quad x \to 0, \tag{18}$$

$$B = \frac{2\pi}{3^{(4\beta+5)/6}} \frac{\Gamma\left(\frac{\beta-1}{3}\right)}{\sin\left(\frac{2\beta+1}{6}\pi\right)\Gamma\left(\frac{\beta}{3}\right)\Gamma\left(\frac{\beta+1}{3}\right)} A. \tag{19}$$

For $0<\beta<1$, i.e., $\frac{1}{2}<r<1$, the leading singular contribution to $P(x)=P(1-x)$ in Eq. (18) vanishes as x approaches 0 or 1, and $P(0)$ is finite and nonzero. For $\beta>1$ or $r<\frac{1}{2}$, $P(x)$ diverges according to Eq. (18) as x approaches 0 or 1. Since the divergence is integrable, $P(x,v)$ can be normalized so that

$$\int_{-\infty}^{\infty} dv \int_0^1 dx\ P(x,v) = \int_0^1 dx\ P(x) = 1 \tag{20}$$

for all $0<r<1$. Note the absence of any special behavior in Eqs. (18) and (19) at $\beta=2$ or $r=r_c$.

The probability density $P(x)$ is shown for several values of r above and below r_c in Fig. 2. The curves were obtained by integrating Eq. (9) over v analytically and then performing the u integration numerically, using the numerical solution for $P(0,u)$ in Fig. 1. Again there is no qualitative difference above and below r_c.

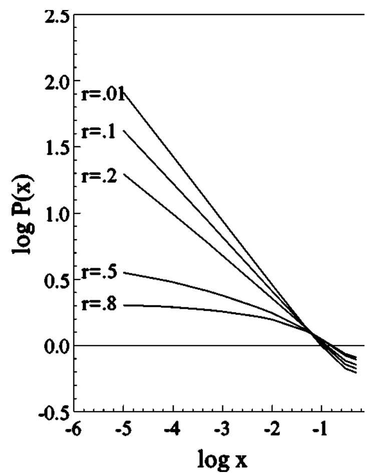

FIG. 2. Double-logarithmic plot (base 10) of $P(x)$ for several values of r. The curves are normalized according to Eq. (20).

III. COLLISION RATE AND MOMENTS OF THE REFLECTED VELOCITY

Unlike the distribution functions $P(x,v)$ and $P(x)$ considered thus far, the equilibrium collision rate I, defined by Eq. (8), does indeed change nonanalytically as r passes through r_c. According to the asymptotic forms (14)–(16) of $P(0,v)$, the second integral on the right of Eq. (8) converges at the upper limit for all $0<r<1$ and at the lower limit for $\beta<2$ but not $\beta \geqslant 2$. Thus the boundary collision rate is finite for $r>r_c$ and infinite for $r \leqslant r_c$, in agreement with the prediction of CSB [1].

The moments $\overline{|v|^q}$ of the velocity just after reflection from the boundary [5] exhibit a closely related collapse transition. Since $vP(0,v)dv$ is the reflected probability current in the velocity range v to $v+dv$,

$$\overline{|v|^q} = \frac{\int_0^{\infty} dv\ v^{q+1} P(0,v)}{\int_0^{\infty} dv\ v P(0,v)}, \quad r>r_c. \tag{21}$$

The denominator in Eq. (21) equals the collision rate I, just shown to be finite for $r>r_c$ and infinite for $r<r_c$. In the latter case, we use the regularized average

$$\overline{|v|^q} = \lim_{\lambda \to 0} \frac{\int_{\lambda}^{\infty} dv\ v^{q+1} P(0,v)}{\int_{\lambda}^{\infty} dv\ v P(0,v)}, \quad r<r_c. \tag{22}$$

From Eqs. (21), (22), and the asymptotic form (14), (15) of $P(0,v)$ for small v, one sees that all the moments $\overline{|v|^q}$ with $q>0$ collapse at $r=r_c$. For $r>r_c$ they are finite and nonzero, and for $r<r_c$ they vanish.

CSB [1] analyzed the case of a randomly accelerated particle, initially at $x=0$ with $v_0>0$, moving on the half-line $x>0$ with inelastic collisions at $x=0$. Defining $Q_n(v,v_0)dv$

as the probability of a velocity just after the nth reflection between v and $v+dv$, normalized so that

$$\int_0^\infty dv\ Q_n(v,v_0) = 1, \tag{23}$$

they calculated $Q_n(v,v_0)$ and the moments

$$\overline{|v_n|^q} = \int_0^\infty dv\ v^q Q_n(v,v_0) \tag{24}$$

exactly. In the limit $n\to\infty$, the qth moment diverges, independent of r, for $q>\frac{1}{2}$. For $0<q<\frac{1}{2}$, this same quantity diverges for $r>r^*(q)$ and vanishes for $r<r^*(q)$. The critical parameter $r^*(q)$, given by Eq. (15) with the replacement $\beta\to q+2$, decreases monotonically from r_c to 0 as q increases from 0 to $\frac{1}{2}$. Thus, in both the semi-infinite geometry $x>0$ and the finite geometry $0<x<1$, certain moments of the reflected velocity collapse as r decreases. However, since boundary collisions are less frequent in the semi-infinite geometry, the velocity fluctuations are greater, and the collapse is less complete. In the semi-infinite case the uncollapsed moments are infinite, the moments with $q>\frac{1}{2}$ do not collapse, and for $0<q<\frac{1}{2}$ the critical parameter $r^*(q)$ is less than r_c.

For a particle confined to $x<0<1$ rather than $x>0$, the recurrence relation that determines $Q_n(v,v_0)$ is given by

$$rQ_{n+1}(rv,v_0) = \int_0^\infty du\ vG(1,v,u)Q_n(u,v_0), \tag{25}$$

$$Q_0(v,v_0) = \delta(v-v_0), \tag{26}$$

as shown in the Appendix. The kernel $G(1,v,u)$ is the same as in Eqs. (12) and (13). Due to the property (A2) (see the Appendix) of the kernel, the recurrence relation preserves the normalization (23). In the limit $n\to\infty$, Eq. (25) becomes identical with the integral equation (12) for $vP(0,v)$, suggesting that $Q_\infty(v,v_0)$ is proportional to $vP(0,v)$.

This proportionality could have been anticipated. In the limit $n\to\infty$, $Q_n(v,v_0)$ is expected to approach the equilibrium distribution $Q_{\text{equil}}(v)$, and $Q_{\text{equil}}(v)$ proportional to $vP(0,v)$ follows from the interpretation of $vP(0,v)dv$ as the reflected probability current, in equilibrium, in the range v to $v+dv$.

The proportionality constant is fixed by the normalization (23). This leads to

$$Q_{\text{equil}}(v) = \frac{vP(0,v)}{\displaystyle\int_0^\infty dv\ vP(0,v)}, \quad r>r_c. \tag{27}$$

For $r<r_c$, the denominator in Eq. (27), which equals the collision rate I, diverges. Regularizing as in Eq. (22), we replace the right side of Eq. (27) by $\lim_{\lambda\to 0}Q(v,\lambda)$, where

$$Q(v,\lambda) = \frac{\theta(v-\lambda)vP(0,v)}{\displaystyle\int_\lambda^\infty dv vP(0,v)}, \tag{28}$$

and $\theta(x)$ denotes the standard step function. Since $\int_0^\infty dv\ Q(v,\lambda)=1$, and since $Q(v,\lambda)$ vanishes in the limit $\lambda\to 0$ except at $v=0+$, where it diverges,

$$Q_{\text{equil}}(v) = \lim_{\lambda\to 0} Q(v,\lambda) = \delta(v), \quad r<r_c. \tag{29}$$

The distribution function $Q_{\text{equil}}(v)$ collapses from Eq. (27) to (29) as r is lowered through r_c. The vanishing of the moments $\overline{|v|^q}=0$, $q>0$ for $r<r_c$ is consistent with the collapsed form (29).

That $Q_{\text{equil}}(v)$ in Eq. (27) is indeed a stationary solution of the recurrence relation (25) follows directly from the integral equation (12) satisfied by $P(0,v)$. That the δ function (29) is a stationary solution for any r may be shown by substituting $\delta(u-\epsilon)$, $\epsilon>0$ on the right side of Eq. (25), integrating over u, and then taking the limit $\epsilon\to 0$.

IV. CLOSING REMARKS

A. Is there inelastic collapse?

The paper of CSB [1] on inelastic collapse is almost entirely concerned with establishing that on the half-line $x>0$ (i) the particle makes an infinite sequence of boundary collisions in a finite time for $r<r_c$, and (ii) in the limit $n\to\infty$ the reflected velocity distribution $Q_n(v,v_0)$ and certain moments of the reflected velocity collapse as r is lowered. Our results for a particle in equilibrium on the finite line $0<x<1$ are quite compatible with (i) and (ii). We question only the statement, below Eq. (19) of [1], that after undergoing an infinite sequence of collisions the particle *remains at rest on the boundary*.

Unlike the central quantity $Q_n(v,v_0)$ in the work of CSB, the solution $P(x,v)$ of the Fokker-Planck equations (5)–(7) provides information on both the position and velocity of the particle in equilibrium. The solution $P(x,v)$ that we have obtained does not collapse onto the boundaries $x=0$ and $x=1$ as r is lowered between 1 and 0. However, for $r<r_c$, $P(0,v)$ diverges more strongly than v^{-2} in the limit $v\to 0$, and this implies $I=\infty$, $\overline{|v|^q}=0$, $q>0$, and $Q_{\text{equil}}(v)=\delta(v)$, via Eqs. (8), (22), and (29). There is a collapse transition in the distribution of reflected velocities $Q_{\text{equil}}(v)$, but it does not involve localization of the particle at the boundaries.

Why is $Q_{\text{equil}}(v)=\delta(v)$ not a sufficient condition for inelastic collapse? Since the velocity $v=0$ on reflection from the boundary is overwhelmingly favored, does not the particle remain at the boundary? In our opinion, the relevant quantity in the question of localization is not $Q_{\text{equil}}(v)$ but the probability per unit time $vP(0,v)dv$ for leaving the boundary with a velocity between v and $v+dv$, where

$$vP(0,v) = IQ_{\text{equil}}(v), \tag{30}$$

as in Eqs. (27) and (28). If $vP(0,v)>0$ for $v>0$, the particle does not remain at the boundary.

For $r < r_c$, the collision rate I is infinite, and for $v > 0$ the product $IQ_{\mathrm{equil}}(v) = I\delta(v)$ on the right side of Eq. (30) is indeterminate. Whether or not $vP(0,v)$ vanishes for $v > 0$ is unclear from Eq. (30). We have calculated $P(0,v)$ for $r < r_c$ by solving the Fokker-Planck equation. The result, as described above, is a smooth function of v with the asymptotic forms (14)–(16). The quantity $vP(0,v)$ does not vanish for $v > 0$, although it does indeed imply $Q_{\mathrm{equil}}(v) = \delta(v)$. Thus, we find that inelastic collisions do not localize the particle at the boundaries.

Below we comment on two earlier results in view of these conclusions.

B. Collision rate in simulations

In computer simulations [2–4] with a discrete time step Δt, the boundary collision rate I, which can never exceed one collision per time step, is necessarily finite. In the algorithm of [3,4], the root-mean-square velocity change is given by $\Delta v = (2\Delta t)^{1/2}$. In the limit $\Delta t \to 0$, the discrete dynamics approaches the continuum dynamics of Eq. (1), and I diverges for $r \leqslant r_c$. Anton [3] has found that the collision rate in his simulations scales as $I \sim (\Delta t)^{(2-\beta)/2}$, $\Delta t \to 0$ for $r < r_c$ and offered a dynamical explanation. We note that this scaling relation follows very simply from our results for the equilibrium distribution function $P(x,v)$. For velocities $|v| \lesssim \Delta v$, the simulation results are expected to deviate from the asymptotic form $P(0,v) \sim v^{-\beta(r)}$ in Eqs. (14) and (15). Thus the boundary collision rate (8) in the simulations scales as

$$I \sim \int_{\Delta v}^{\infty} dv\ v^{1-\beta} \sim (\Delta v)^{2-\beta} \sim (\Delta t)^{(2-\beta)/2}, \quad \Delta t \to 0. \tag{31}$$

C. Persistence exponent for $r < r_c$

Burkhardt [10] and De Smedt *et al.* [11] have considered the probability $Q(x_0, v_0, t)$ that a randomly accelerated particle with initial position and velocity x_0, v_0, confined to the half-line $x > 0$ and reflected inelastically at $x = 0$, has not yet *undergone inelastic collapse* after a time t. They predicted $Q(x_0, v_0, t) = 1$ for $r > r_c$, and for $r < r_c$ the power-law decay

$$Q(x_0, v_0; t) \sim t^{(2-\beta)/2}, \quad t \to \infty, \tag{32}$$

where the exponent β is the same as in Eqs. (14), (15), and (31). In view of our conclusions that the particle makes an infinite number of collisions in a finite time but does not remain at the boundary, $Q(x_0, v_0, t)$ in Eq. (32) should be interpreted as the probability that after a time t the randomly accelerated particle has not yet made an infinite number of boundary collisions. The derivations of Eq. (32) in [10,11] are compatible with this interpretation, and it is also supported by simulations [3,12,13].

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APPENDIX A: VELOCITY DISTRIBUTION ON ARRIVAL AT THE BOUNDARY

The probability that a randomly accelerated particle with initial position $x = 0$ and initial velocity $u > 0$, moving on the half-line $x > 0$, arrives with speed between v and $v + dv$ on its first return to $x = 0$ is given by $vG_0(v,u)dv$, where

$$G_0(v,u) = \frac{3}{2\pi} \frac{v^{1/2}u^{1/2}}{v^3 + u^3}. \tag{A1}$$

This result, due to McKean [8], was also obtained independently by CSB [1].

The quantity $G(1,v,u)$ in Eq. (13), derived by BFG [4], extends this result to the finite interval $0 < x < 1$. The probability that a randomly accelerated particle which leaves $x = 0$ with velocity $u > 0$ has speed between v and $v + dv$ the next time it reaches either boundary is given by $vG(1,v,u)dv$, where the first and second terms on the right side of Eq. (13) correspond to arrival at $x = 0$ and $x = 1$, respectively. Like $G_0(v,u)$ in Eq. (A1), $G(1,v,u)$ satisfies the normalization condition

$$\int_0^{\infty} dv\ vG(1,v,u) = 1. \tag{A2}$$

Integral equation (12) for $P(0,v)$ follows directly from the interpretation of $G(1,v,u)$ in the preceding paragraph and the stationarity, in equilibrium, of the reflected current $vP(0,v)dv$ between v and $v + dv$. Another consequence is the recurrence relation (25) for the probability distribution $Q_n(v, v_0)$ of the speed with which the particle rebounds after the nth boundary collision. Solving Eqs. (25) and (26) with $G_0(v,u)$ in Eq. (A1) in place of $G(1,v,u)$, CSB [1] calculated $Q_n(v, v_0)$ exactly for motion on the half-line $x > 0$.

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